PMM U.S.S.R., Vol.44,pp.151-157
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0021-8928/81/2 0151 $7.50/0
UDC 531.36

ON FIRST APPROXIMATION STABILITY RELATIVE TO A PART OF VARIABLES*

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Certain theorems on stability in the first approximation relative to a part of variables which generalize the Liapunov and Massera theorems, are proved.

1. The case of regular linear approximation system. 1°. We consider the nonlinear system

$$dx/dt = M(t)x + F(t,x),\ x = \mathrm{col}(x_1,\ldots,x_n)$$
$$F(t,x) = \mathrm{col}(F_1(t,x),\ldots,F_n(t,x)),\ F(t,0)\equiv 0 \tag{1.1}$$
$$M(t)\in C[t_0,\infty),\quad \sup_t \|M(t)\| < \infty$$

where $M(t)$ is the lower triangular matrix of order n and $\|\cdot\|$ is the Euclidean norm.

We shall investigate this system stability with respect to variables $x_1,\ldots,x_m$ $(1\leqslant m\leqslant n)$, using the notation

$$y = \mathrm{col}(y_1,\ldots,y_m),\quad y_k = x_k\quad (k=1,\ldots,m)$$
$$z = \mathrm{col}(z_1,\ldots,z_p),\quad z_k = x_{k+m}\quad (k=1,\ldots,p=n-m)$$
$$f(t,y,z) = \mathrm{col}(F_1(t,x),\ldots,F_m(t,x))$$
$$g(t,y,z) = \mathrm{col}(F_{m+1}(t,x),\ldots,F_n(t,x))$$
$$M(t) = \begin{Vmatrix} A(t) & 0 \\ C(t) & B(t)\end{Vmatrix}$$

where $A(t)$ and $B(t)$ are lower triagular matrices of order $m\times m$ and $p\times p$, respectively.

System (1.1) can now be represented in the form

$$dy/dt = A(t)y + f(t,y,z)$$
$$dz/dt = C(t)y + B(t)z + g(t,y,z) \tag{1.2}$$

Let us assume that

a) the vector function $F(t,x)$ is continuous and satisfies the conditions of uniqueness of solution in the region

$$t\geqslant t_0,\ \|y\| < H\ (H>0),\ 0\leqslant \|z\| < \infty$$

b) solutions of system (1.1) are z-continuable which means that any solution $x(t)$ is determined for all $t\geqslant t_0$ for which $\|y\| < H$.

We denote by $x = x(t;t_0,x_0)$ the solution of system (1.1) determined by the initial condition $x(t_0;t_0,x_0) = x_0$.

Together with (1.2) we shall consider the linear system

$$dy^*/dt = A(t)y^* \tag{1.3}$$

Theorem 1. If

1) the linear system (1.3) is Liapunov regular,
2) all characteristic indices of system (1.3) are negative,

$$\alpha_1\leqslant\alpha_2\leqslant\ldots\leqslant\alpha_m<0 \quad \text{and}$$

3) the vector function f satisfies the inequality

$$\|f(t,y,z)\|\leqslant\psi(t)\|y\|^q\quad (q>1) \tag{1.4}$$

in which $\psi(t)$ is a continuous positive function in $[t_0,\infty)$, and $\chi[\psi(t)]=0$, then the trivial solution $x\equiv 0$ of system (1.1) is exponentially y-stable as $t\to\infty$.

Proof. Let $\alpha_m < -\gamma < 0$. We apply to system (1.1) the transform

$$x = we^{-\gamma(t-t_0)} \tag{1.5}$$

and obtain

$$dw/dt = N(t)w + G(t,w)$$
$$N(t) = \gamma E + M(t),\quad G(t,w) = e^{\gamma(t-t_0)}F(t, we^{-\gamma(t-t_0)}) \tag{1.6}$$

As the result of transformation (1.5), system (1.2) assumes the form

*Prikl.Matem.Mekhan.,44,No.2,211-220, 1980

$$\begin{aligned} du/dt &= A_1(t)\,u + f_1(t,u,v) \\ dv/dt &= C_1(t)\,u + B_1(t)\,v + g_1(t,u,v) \\ u &= ye^{\gamma(t-t_0)}, \quad v = ze^{\gamma(t-t_0)} \\ u &= \mathrm{col}\,(u_1,\dots,u_m); \quad u_k = w_k \ (k = 1,\dots,m) \\ v &= \mathrm{col}\,(v_1,\dots,v_p); \quad v_k = w_{k+m} \quad (k = 1,\dots,p) \\ A_1 &= \gamma E + A,\ B_1 = \gamma E + B,\ C_1 = C \end{aligned} \tag{1.7}$$

Obviously $A_1(t)$ and $B_1(t)$ are also lower triangular matrices and

$$\begin{aligned} f_1(t,u,v) &= e^{\gamma(t-t_0)} f(t, ue^{-\gamma(t-t_0)}, ve^{-\gamma(t-t_0)}) \\ g_1(t,u,v) &= e^{\gamma(t-t_0)} g(t, ue^{-\gamma(t-t_0)}, ve^{-\gamma(t-t_0)}) \end{aligned} \tag{1.8}$$

Moreover, $w(t_0) = x(t_0)$, and $G(t,w)$ satisfies conditions a) and b), i.e. transform (1.5) preserves the existence of the unique solution and, also, the z-continuation of solutions.

System

$$du^*/dt = A_1(t)\,u^* \tag{1.9}$$

is obviously regular.

Let $H(t)\,(H(t_0) = E)$ be the fundamental lower triangular matrix of the system

$$dw^*/dt = N(t)\,w^* \tag{1.10}$$

Applying the method of variation of constants, we replace the nonlinear differential equation by the equivalent integral equation

$$\begin{aligned} w(t) &= H(t)\,w(t_0) + \int_{t_0}^{t} K(t,\tau)\,G(\tau, w(\tau))\,d\tau \\ K(t,\tau) &= H(t)\,H^{-1}(\tau), \quad w(t_0) = \mathrm{col}\,(u(t_0), v(t_0)) = x(t_0) \end{aligned} \tag{1.11}$$

Since $H(t)$ is a lower triangular matrix, $K(t,\tau)$ is of the same form.

In conformity with the local theorem of existence of solutions there exists for the pair (t_0, w_0), where $\|u_0\| < H$, the solution $w(t)$ of the differential equation (1.6), which satisfies the initial condition $w(t_0) = x(t_0)$, and is determinate in some interval $t_0 \leqslant t < t_0 + l$, and $\|u(t)\| < H$ for $t \in [t, t_0 + l)$.

Let $H(t)$ and $K(t,\tau)$ be of the form

$$H(t) = \begin{Vmatrix} H_1(t) & 0 \\ H_3(t) & H_2(t) \end{Vmatrix}, \quad K(t,\tau) = \begin{Vmatrix} K_1(t,\tau) & 0 \\ K_3(t,\tau) & K_2(t,\tau) \end{Vmatrix}$$

where H_1, K_1 and H_2, K_2 are lower triangular matrices of order $m \times m$ and $p \times p$, respectively. Then, in conformity with (1.11), the vector function $u(t)$ satisfies the integral equation

$$u(t) = H_1(t)\,u(t_0) + \int_{t_0}^{t} K_1(t,\tau)\,f_1(\tau, u(\tau), v(\tau))\,d\tau \tag{1.12}$$

which yields the estimate

$$\|u(t)\| \leqslant \|H_1(t)\|\,\|u(t_0)\| + \int_{t_0}^{t} \|K_1(t,\tau)\|\,\|f_1(\tau, u(\tau), v(\tau)\|\,d\tau \tag{1.13}$$

Since all characteristic indices $\beta_k = \alpha_k + \gamma$ of the linear system (1.9) are negative, there exists a number $c_1 \geqslant 1$ such that

$$\|H_1(t)\| < c_1 \quad \text{for } t_0 \leqslant t < \infty \tag{1.14}$$

Moreover, on the strength of estimate of the Cauchy matrix or the regular system with negative characteristic exponents /2/ we have

$$\|K_1(t,\tau)\| < c_2 e^{\varepsilon(\tau - t_0)} \quad \text{for } t_0 \leqslant \tau \leqslant t < \infty \tag{1.15}$$

On the basis of formulas (1.4) and (1.8) we have

$$\|f_1(t,u,v)\| = e^{-\gamma(t-t_0)}\,\|f(t, ue^{-\gamma(t-t_0)}, ve^{-\gamma(t-t_0)})\| < c_3 \exp\{[\varepsilon - (q-1)\gamma](t - t_0)\}\,\|u\|^q \tag{1.16}$$

where c_3 is a fairly large positive number.

Substituting (1.14)—(1.16) into (1.13) we obtain the estimate

$$\|u(t)\| < c_1\|u(t_0)\| + \int_{t_0}^{t} c_2 c_3 \exp\{[2\varepsilon - (q-1)\gamma](\tau - t_0)\}\|u(\tau)\|^q d\tau\,, \qquad t_0 \leqslant t < t_0 + l \tag{1.17}$$

We select the positive number ε so small as to satisfy the inequality

$$\delta = (q-1)\gamma - 2\varepsilon > 0$$

Then applying to the inequality

$$\|u(t)\| \leqslant c_1 \|u(t_0)\| + \int_{t_0}^{t} c_4 e^{-\delta(\tau - t_0)} \|u(\tau)\|^q d\tau \quad (c_4 = c_2 c_3) \tag{1.18}$$

the Bihari lemma /2/, we concluded that

$$\|u(t)\| \leqslant c_1 \|u(t_0)\| \, [1 - Q(t)]^{-1/(q-1)} \tag{1.19}$$

$$Q(t) = (q-1) c_1^{q-1} \|u_1(t_0)\|^{q-1} \int_{t_0}^{t} c_4 e^{-\delta(\tau - t_0)} d\tau$$

if only

$$Q(t) < 1 \tag{1.20}$$

Since

$$\int_{t_0}^{t} e^{-\delta(\tau - t_0)} d\tau < \frac{1}{\delta} < \infty$$

then, provided that $\|u(t_0)\| = \|y(t_0)\|$ is fairly small, it is always possible to assume that inequality (1.20) is satisfied. It follows from (1.19) that when $\|u(t_0)\|$ is fairly small, then for any $t \in [t_0, t_0 + l)$ $u(t)$ is an inner point of region $\{t_0 \leqslant t < \infty, \|u\| \leqslant H/2 < H\}$ and, consequently, the solution $w(t)$ is infinitely u-continuable to the right. The solution $w(t)$ is by virtue of assumption b) infinitely continuable to the right. Thus for $t_0 \leqslant t < \infty$ we have the inequality

$$\|u(t)\| \leqslant L \|y_0\| < H/2$$

where L is some constant dependent on t_0.

Reverting to the variable x, with $t_0 \leqslant t < \infty$ and $\|y(t_0)\| < \Delta < H$ (H fairly small) we have

$$\|y(t)\| \leqslant L \|y(t_0)\| e^{-\gamma(t-t_0)} \leqslant L(\|y(t_0)\| + \|z(t_0)\|) e^{-\gamma(t-t_0)}$$

i.e. the trivial solution $x \equiv 0$ of the nonlinear system (1.1) is exponentially y-stable as $t \to \infty$. The theorem is proved.

2°. Let us consider a nonlinear system of the more general form

$$dx/dt = M(t)x + F(t, x); \; M(t) \in c\,[t_0, \infty), \; \sup_t \|M(t)\| < \infty \tag{1.21}$$

in which $M(t)$ is an $(n \times n)$ matrix, the vector function $F(t, x)$ conforms to assumptions a) and b), and $F(t, 0) \equiv 0$. We use here in addition to the notation introduced above the following:

$$P_k x = \mathrm{col}(x_1, \ldots, x_k) \quad (1 \leqslant k \leqslant n)$$

$$X_k = [x^{(1)}, \ldots, x^{(k)}], \quad X_{n-k} = [x^{(k+1)}, \ldots, x^{(n)}]$$

where $G(X_k)$ is the Gram determinant composed of vectors $x^{(1)}, \ldots, x^{(k)}$.

Theorem 2. Let

1) for the linear approximation system

$$dx^*/dt = M(t)x^* \tag{1.22}$$

of system (1.21) exist a normal basis $X^* = [x^{*(1)}(t), \ldots, x^{*(n)}(t)]$ such that

$$\inf_t \frac{G(X^*)}{G(X_m^*)\, G(X_{n-m}^*)} = \rho > 0 \tag{1.23}$$

2) the linear system (1.22) be Liapunov regular,

3) the characteristic indices of vectors $x^{*(1)}(t), \ldots, x^{*(m)}(t)$ be negative

$$\chi[x^{*(i)}(t)] = \alpha_i < 0 \quad (i = 1, \ldots, m) \tag{1.24}$$

4) for the vector function $F(t, x)$ the inequality

$$\|f(t, y, z)\| \leqslant \psi(t) \|y\|^q \tag{1.25}$$

where $\psi(t)$ is a continuous positive function in $B[t_0, \infty)$ be satisfied, and

$$\chi[\psi(t)] = 0 \tag{1.26}$$

The trivial solution $x \equiv 0$ of system (1.21) is, then, exponentially y-stable as $t \to \infty$.

Proof. Condition (1.23) implies the existence of the Liapunov transform $x^* = U(t)\xi^*$ which converts system (1.22) into a partitioned lower triangular system (see /1/)

$$d\xi^*/dt = Q(t)\,\xi^* \tag{1.27}$$

with this transformation the nonlinear system (1.21) assumes the form

$$\begin{aligned} d\xi/dt &= Q(t)\,\xi + G(t, \xi) \\ Q(t) &= U^{-1}(t)\,M(t)\,U(t) - U^{-1}(t)\,U^{\cdot}(t) \\ G(t, \xi) &= U^{-1}(t)\,F(t, U(t)\,\xi) \end{aligned} \tag{1.28}$$

or

$$\begin{aligned} d\eta/dt &= A(t)\,\eta + h(t, \eta, \zeta) \\ d\zeta/dt &= B(t)\,\zeta + h_1(t, \eta, \zeta) \end{aligned} \tag{1.29}$$

where $A(t)$ and $B(t)$ are lower triangular matrices of order m and $p = n - m$, respectively, η is an m-dimensional vector, ζ is a p-dimensional vector, $\xi = \mathrm{col}(\eta, \zeta)$, and $G(t, \xi) = \mathrm{col}(h(t, \eta, \zeta), h_1(t, \eta, \zeta))$.

Since system (1.22) is regular, system (1.27) must also be regular. By the criterion of regularity of a triangular system /2/ the linear system

$$d\eta^*/dt = A(t)\,\eta^*$$

is regular.

Since the Liapunov transform does not alter the characteristic indices, hence

$$\chi[\eta^{*(i)}] = \chi[\xi^{*(i)}] = \chi[x^{*(i)}] = \alpha_i < 0 \quad (i = 1, \ldots, m)$$

Taking into consideration the boundedness of matrices $U(t)$ and $U^{-1}(t)$, from formulas (1.25) and (1.29) we obtain

$$\| h(t, \eta, \zeta) \| \leqslant \psi_1(t)\, \| \eta \|^q \quad (q > 1)$$

where $\psi_1(t)$ is a continuous function positive in $[t_0, \infty)$ which satisfies equality (1.26). It is, moreover, evident that $G(t, \xi)$ conforms to assumptions of the a) and b) type.

All conditions of Theorem 1 have been, thus, satisfied for system (1.29) and, consequently, the trivial solution $\xi \equiv 0$ of that system is exponentially η-stable as $t \to \infty$, i.e.

$$\| \eta(t) \| \leqslant L\,(\| \eta(t_0) \| + \| \zeta(t_0) \|)\, e^{-\gamma(t-t_0)}$$

where L is a constant, the quantity $\| \eta(t_0) \|$ fairly small, and $\alpha_i < -\gamma < 0 \quad (i = 1, \ldots, m)$.

Since $x = U(t)\,\xi$, hence

$$\| y(t) \| \leqslant \| U(t) \|\, \| \eta(t) \| \leqslant L_1 (\| y(t_0) \| + \| z(t_0) \|)\, e^{-\gamma(t-t_0)}$$

which means that the solution $x \equiv 0$ of system (1.21) is exponentially y-stable as $t \to \infty$. The theorem is proved.

2. The case of the irregular system of linear approximations. Let us consider in this case the problem of first approximation stability relative to a part of variables.

1°. Consider the differential system

$$\begin{aligned} dx/dt &= M(t)\,x + F(t, x) \\ M(t) &\in C[t_0, \infty), \quad \sup_t \| M(t) \| < \infty \end{aligned} \tag{2.1}$$

where $M(t)$ is the lower triangular matrix, $F(t, x)$ conforms to assumptions a) and b) from Sect. 1, and $F(t, 0) \equiv 0$.

System (2.1) may be written in the form

$$\begin{aligned} dy/dt &= A(t)\,y + f(t, y, z) \\ dz/dt &= C(t)\,y + B(t)\,z + g(t, y, z) \end{aligned} \tag{2.2}$$

Theorem 3. Let

1) the inequality

$$\| f(t, y, z) \| \leqslant \psi(t)\, \| y \|^q \quad (q > 1) \tag{2.3}$$

where $\psi(t)$ is a continuous positive function, be satisfied, and $\chi[\psi(t)] = 0$,

2) the characteristic indices of the linear function

$$dy^*/dt = A(t)\,y^* \tag{2.4}$$

satisfy the condition

$$\alpha_1 \leqslant \ldots \leqslant \alpha_m = \alpha < -\frac{\varkappa}{q-1} \leqslant 0$$

where $\varkappa$, a coefficient of the irregular system (2.4), is defined by

$$\varkappa = \sum_{k=1}^{m} \alpha_k - \varliminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \mathrm{Tr}\, A(t_1)\, dt_1$$

The trivial solution $x \equiv 0$ of the nonlinear system (1.1) (or of system (2.2)) is asymptotically y-stable as $t \to \infty$.

Proof. Let γ be a positive number such that

$$\varkappa/(q-1) < \gamma < -\alpha \tag{2.5}$$

We set

$$D = \operatorname{diag}(\alpha_1 + \gamma, \ldots, \alpha_m + \gamma, 1, \ldots 1) = \operatorname{diag}(D', E)$$

Let $X(t) = [x_{jk}(t)]_n^n$ be a normalized fundamental lower triangular matrix of $(X(t_0) = E)$ of the linear system

$$dx^*/dt = M(t)\, x^* \tag{2.6}$$

It is now obvious that matrix $Y(t) = [y_{jk}(t)]_m^m$ in which $y_{ik}(t) = x_{jk}(t)$ for $j, k = 1, \ldots, m$ is the fundamental matrix of system (2.4) and $Y(t_0) = E$.

We apply to system (2.1) the transform

$$x = X(t)\, e^{-Dt} w, \quad w = \operatorname{col}(u, v)$$

where u is an m-dimensional vector and v an $(n-m)$-dimensional vector, and obtain

$$\frac{dx}{dt} = X(t)\, e^{-Dt} \frac{dw}{dt} + X^{\cdot}(t)\, e^{-Dt} w - X(t)\, e^{-Dt} D w = M(t)\, X(t)\, e^{-Dt} w + F(t, X(t)\, e^{-Dt} w)$$

from which follows that

$$dw/dt = Dw + e^{Dt} X^{-1}(t)\, F(t, X(t) e^{-Dt} w) \tag{2.7}$$

Since $X(t)$ is a lower triangular matrix, $X^{-1}(t)$ and $e^{Dt}X^{-1}(t)$ are also lower triangular matrices. Furthermore, it follows from (2.7) that

$$du/dt = D'u + h(t, u, v) \tag{2.8}$$

$$h(t, u, v) = P_m\, [e^{Dt} X^{-1}(t)\, F(t, X(t)\, e^{-Dt} w)]$$

It is known that

$$Y^{-1} = \frac{1}{\Delta(t)} \| \Delta_{kj}(t) \|, \quad \Delta(t) = \det Y(t)$$

where $\Delta_{kj}(t)$ is the cofactor of the determinant $\Delta(t)$. Using the Ostrogradskii — Liouville formula and taking into account the equality $\Delta(t_0) = 1$, we obtain

$$\Delta(t) = \exp \int_{t_0}^{t} \operatorname{Tr} A(t_1)\, dt_1$$

Hence

$$Y^{-1}(t) = \left\| \Delta_{kj}(t) \exp\left[-\int_{t_0}^{t} \operatorname{Tr} A(t_1)\, dt_1\right] \right\|$$

from which

$$\chi[e^{D't} Y^{-1}(t)] = \chi\left[e^{(\alpha_j+\gamma)t} \Delta_{kj}(t) \exp\left[-\int_{t_0}^{t} \operatorname{Tr} A(t_1)\, dt_1\right]\right] \leqslant \tag{2.9}$$

$$\max_{j,k}\left[\alpha_j + \gamma + \sum_k \alpha_k - \alpha_j - \varliminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{Tr} A(t_1)\, dt_1\right] = \varkappa + \gamma$$

$$\chi[Y(t)\, e^{-D't}] = \chi\, \{[y_{jk} e^{-(\alpha_k+\gamma)t}]\} \leqslant \max_{j,k} [\alpha_k - (\alpha_k + \gamma)] = -\gamma < 0$$

Since $\chi[Y(t)\, e^{-D't}] < 0$, hence $Y(t)\, e^{-D't} \to 0$ as $t \to \infty$.

Let $\|u\| \leqslant H/M$. Then

$$\|y\| \leqslant \|Y(t)\, e^{-D't}\| \, \|u\| \leqslant H$$

Let us evaluate the nonlinear term $h(t, u, v)$ of system (2.8) with $\|u\| \leqslant H/M$. Using inequality (2.3) we obtain

$$\|h(t, u, v)\| \leqslant \|P_m (e^{Dt}\, X^{-1}(t)\, F(t, X(t)\, e^{-Dt} w))\| \leqslant$$

$$\|e^{D't} Y^{-1}(t)\| \, |\psi(t)| \, \|Y(t)\, e^{-D't}\|^q \|u\|^q = \varphi(t) \|u\|^q$$

$$\varphi(t) = \|e^{D't} Y^{-1}(t)\| \, |\psi(t)| \, \|Y(t)\, e^{-Dt}\|^q$$

By virtue of inequality (2.9) and properties of characteristic indices the following estimate is valid:

$$\chi[\varphi(t)] = \chi\, [\|e^{D't} Y^{-1}(t)\| \, |\psi(t)| \, \|Y(t)\, e^{-D't}\|^q] \leqslant \varkappa + \gamma + 0 - q^\gamma = \varkappa - (q-1)^\gamma$$

On the basis of inequality (2.5) we have

$$\chi\,[\varphi\,(t)] < 0$$

hence

$$\|\,h\,(t,\,u,\,v)\,\| \leqslant C\,\|\,u\,\|^{q}, \quad q > 1$$
$$(t_0 \leqslant t < \infty,\ \|\,u\,\| \leqslant H\,/\,M)$$

Thus by virtue of the Liapunov theorem on the stability of a quasilinear system /2/ the trivial solution $u \equiv 0$ of system (2.8) is asymptotically stable as $t \to \infty$. This means that the solution $w \equiv 0$ of system (2.7) is asymptotically u-stable as $t \to \infty$. From this on the basis of formulas $y = Y\,(t)\,\exp\,(-D't)u$ and $\chi\,[Y\,(t)\,\exp\,(-D't)] < 0$ follows that the trivial solution of system (1.1) is asymptotically y-stable as $t \to \infty$. Theorem 3 is proved.

2°. Let us now consider a differential system of a more general form

$$dx\,/\,dt = M\,(t)x + F\,(t,\,x)$$

where $(M\,(t)$ is an $(n \times n)$ matrix.

We shall prove for this general system a theorem similar to Theorem 3.

First of all we shall prove two lemmas.

Let us consider the linear homogeneous system of order n

$$dx\,/\,dt = S\,(t)\,x,\ S\,(t) \in C\,[t_0,\,\infty),\ \sup_t \|\,S\,(t)\,\| < \infty \tag{2.10}$$

assuming it to be irregular with

$$\varkappa = \sum_{k=1}^{n} \alpha_k - \varliminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \mathrm{Re\,Tr}\,S\,(t_1)\,dt_1 \tag{2.11}$$

as its coefficient of irregularity in which $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_n$ is the complete spectrum of system (2.10).

By applying to system (2.10) the Liapunov transform $x = U\,(t)\,y$ we convert it to the form

$$dy/dt = Q\,(t)\,y \tag{2.12}$$
$$Q\,(t) = U^{-1}\,(t)\ S\,(t)\,U\,(t) - U^{-1}\,(t)\,U^{\cdot}\,(t)$$

We denote the complete spectrum of system (2.12) by $\alpha_1' \leqslant \alpha_2' \leqslant \ldots \leqslant \alpha_n'$ and its coefficient of irregularity by $\varkappa'$.

Lemma 1. The Liapunov transform preserves the irregularity coefficient of a linear homogeneous system of the form (2.10), i.e. $\varkappa = \varkappa'$.

Proof of this lemma follows directly from that the Liapunov transform retains the characteristic indices and value of the limit in formula (2.11).

We call the number

$$\varkappa\,(m) = \sum_{k=1}^{m} \alpha_k - \varliminf_{k\to\infty} \frac{1}{t} \int_{t_0}^{t} \mathrm{Re} \sum_{i=1}^{m} s_i\,(t_1)\,dt_1$$

where s_i are diagonal elements of matrix $S\,(t)$, the coefficient of m-partial irregularity of the linear system (2.10).

Lemma 2. The coefficient of partial irregularity of a linear system is retained when the Liapunov transform $x = U\,(t)\,y$ is such that the matrix of transformation is of partitioned diagonal form

$$U\,(t) = \mathrm{diag}\,(U_m,\ U_{n-m})$$

where U_m and U_{n-m} are square matrices of order m and $n - m$, respectively. The proof of this lemma is obvious.

Let us now consider the nonlinear system

$$dx\,/\,dt = M\,(t)\,x + F\,(t,\,x) \tag{2.13}$$
$$M\,(t) \in C\,[t_0,\,\infty),\ \ \sup_t \|\,M\,(t)\,\| < \infty$$

where $(M\,(t)$ is an $(n \times n)$- matrix, $F\,(t,\,x)$ conforms to assumptions a) and b), and $F\,(t,\,0) \equiv 0$.

Using the notation introduced above we can write (2.13) as

$$dy\,/\,dt = A\,(t)\,y + B\,(t)\,z + f\,(t,\,y,\,z) \tag{2.14}$$
$$dz\,/\,dt = C\,(t)\,y + D\,(t)\,z + g\,(t,\,y,\,z)$$

Theorem 4. Let

1) system (2.14) have a normal fundamental matrix of the partitioned diagonal form $U\,(t) = \mathrm{diag}\,(U_m\,(t),\,U_{n-m}\,(t))$ that satisfies the inequality

$$\inf \frac{G(U)}{G(U_m)G(U_{n-m})} = \rho > 0 \tag{2.15}$$

2) the inequality

$$\| f(t, y, z) \| \leqslant \psi(t) \| y \|^q \quad (q > 1)$$

where $\psi(t)$ is a continuous positive function, and $\chi[\psi(t)] = 0$, be valid, and

3) the characteristic indices of the system

$$dy^*/dt = A(t)\, y^*$$

satisfy the condition

$$\alpha_1 \leqslant \ldots \leqslant \alpha_m = \alpha < -\frac{\varkappa}{q-1} \leqslant 0$$

where $\varkappa$ is the coefficient of the m-partial irregularity.

Then the trivial solution $x \equiv 0$ of the nonlinear system (2.13) (or (2.14)) is asymptotically y-stable as $k \to \infty$.

Proof. Condition 1) of the theorem implies that the linear differential system

$$dx^*/dt = M(t)\, x^*$$

can be transformed to a partitioned lower triangular system by applying the Liapunov transform $x^* = U(t)\,\xi^*$, where $U(t) = \operatorname{diag}(U_m, U_{n-m})$.

Let the transformed system be of the form

$$d\xi^*/dt = Q(t)\,\xi^*$$

where $Q(t)$ is a partitioned lower triangular matrix. The nonlinear system (2.13) is then reduced to system

$$d\xi/dt = Q(t)\,\xi + G(t, \xi), \quad \xi = \operatorname{col}(\eta, \zeta),\ Q(t) = \operatorname{diag}(A_1(t), B_1(t)),\ G(t, \xi) = \operatorname{col}(h(t, \eta, \zeta), h_1(t, \eta, \zeta)) \tag{2.16}$$

where η is an m-dimensional vector and ζ is an $(n-m)$-dimensional vector.

On the basis of the Liapunov transform properties and of the lemma we obtain the equalities

$$\varkappa' = \varkappa, \quad \alpha_i' = \alpha_i \quad (i = 1, \ldots, m)$$

where $\varkappa$ is the irregularity coefficient and α_i' are the characteristic indices of the system

$$d\eta^*/dt = A_1(t)\,\eta^*$$

From this

$$\alpha_1' \leqslant \ldots \leqslant \alpha_m' = \alpha < -\frac{\varkappa'}{q-1} \leqslant 0$$

Since matrices $U(t)$ and $U^{-1}(t)$ are bounded and $h(t, \eta, \zeta) = U_m^{-1}(t)\, f(t, U_m^{-1}(t)\,\eta, U_{n-m}^{-1}(t)\,\zeta)$, by virtue of inequality (2.15) we have

$$\| h(t, \eta, \zeta) \| \leqslant \psi_1(t) \| \eta \|^q \quad (q > 1)$$

where $\psi_1(t)$ is a positive function for $t \in [t_0, \infty)$, and $\chi[\psi_1(t)] = 0$.

By virtue of Theorem 3 the trivial solution $\xi \equiv 0$ of system (2.16) is asymptotically ζ-stable as $t \to \infty$.

This shows that the solution $x \equiv 0$ of system (2.13) is asymptotically y-stable as $t \to \infty$. The Theorem 4 is proved.

REFERENCES

1. BYLOV, B. F., VINOGRAD, R. E., GROBMAN, D. M., and NEMYTSKII, V. V., Theory of Liapunov Indices. Moscow, "Nauka", 1966.

2. DEMIDOVICH, B. P., Lectures on the Mathematical Theory of Stability. Moscow, "Nauka",1967.

Translated by J.J.D.